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Solving a coupled-channel cavity QED model using supersymmetric unitary transformation

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Abstract
In this paper, we use supersymmetric unitary transformation theory to solve a coupled-channel cavity QED model which includes the Stark term and frequency detuning. We give the eigenvalue, eigenstates and time evolution of the state vector of the system.

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Supersymmetric quantum mechanics has attracted much attention because it presents a concise and wonderful method for solving the eigenvalue equations of quantum mechanics. Recently, the supersymmetric method has been used effectively to solve the energy level of many potential problems [1–3], In [4] Hong-yi Fan proposed that the Jaynes–Cummings model can be solved by a supersymmetric unitary transformation. There is no doubt that this new method will further enrich the contents of supersymmetric quantum mechanics. In this paper, we use supersymmetric unitary transformation theory to solve a coupled-channel cavity quantum electrodynamics (QED) model which includes the Stark term and the frequency detuning [5]. We give the eigenvalue, eigenstates and time evolution of the state vector of the system.

The Hamiltonian of the fully quantized coupled-channel cavity QED model reads [5]

$$H = \omega_p N + E_{+-} J_z + (gL_z + \delta)(1 - \sigma_z)/2 + g(a_S^+ a_P + a_P^+ a_A)\sigma_+ + g(a_P^+ a_S + a_A^+ a_P)\sigma_- \qquad (1)$$

where the subscripts P, S and A represent pump, Stokes and anti-Stokes modes, respectively; a and a^+ are the creation and annihilation operators for the corresponding modes, σ are the usual atomic transition operators; $L_z = n_A - n_S$, $J_z = L_z + \frac{1}{2}\sigma_z$ and $N = n_P + n_S + n_A$; $gL_z(1-\sigma_z)/2$ is the Stark term and $\delta(1 - \sigma_z)/2$ represents frequency detuning. It can be proved that N and J_z are constants of motion. To determine the energy eigenvalues of the coupled-channel cavity QED model, Wang *et al* have given a connection between field variables and the orbital angular momentum [6],

$$L_+ = \sqrt{2}(a_S a_P^+ + a_A^+ a_P) \tag{2a}$$

$$L_- = \sqrt{2}(a_P a_S^+ + a_P^+ a_A) \tag{2b}$$

$$L_z = a_A^+ a_A - a_S^+ a_S \tag{2c}$$

$$L^2 = (n_A - n_S)^2 + (n_A + n_S)(2n_P + 1) + 2n_P + 2(a_P^2 a_A^+ a_S^+ + a_P^{+2} a_S a_A) \tag{2d}$$

where L_- and L_+ are the lowering and raising operators of the momentum. They succeeded in obtaining the eigenvalues and corresponding eigenvectors for the special case of $J_z = -\frac{1}{2}$ and without a Stark term or frequency detuning. Recently, Ying Wu presented a simple algebraic method to solve this model by establishing its connection with the model for a spin-$\frac{1}{2}$ particle in a magnetic field, and taking advantage of equations (2). He obtained the eigenvalues and corresponding eigenvectors for the special case without Stark term and frequency detuning [5]. He also gave the eigenvalues but not the eigenvectors for the general case described by equation (1). Here, we use supersymmetric unitary transformation theory to solve the general coupled-channel cavity QED model. We give the eigenvalue, eigenvectors and time evolution of the state vector of the system.

To construct the supersymmetric unitary transformation operator, we first define the supersymmetric transformation generators as follows:

$$Q = L_+\sigma_- \qquad Q^+ = L_-\sigma_+ \tag{3a}$$

$$N' = L_+L_-\sigma_{--} + L_-L_+\sigma_{++} = L^2 - J_z^2 + \tfrac{1}{4} \tag{3b}$$

where

$$\sigma_{++} = \sigma_+\sigma_- = \tfrac{1}{2}(1 + \sigma_z) \qquad \sigma_{--} = \sigma_-\sigma_+ = \tfrac{1}{2}(1 - \sigma_z). \tag{4}$$

It is easy to see that (N', Q^+, Q) form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.

$$\begin{aligned} &Q^2 = Q^{+2} = 0 \qquad [Q^+, Q] = N'\sigma_z \qquad \{Q, \sigma_z\} = \left\{Q^+, \sigma_z\right\} = 0 \\ &N' = \left\{Q, Q^+\right\} \qquad [N', Q] = [N', Q^+] = 0 \qquad (Q^+ - Q)^2 = -N' \end{aligned} \tag{5}$$

in which { } denotes the anticommutation bracket. With the help of equations (2) and (3), equation (1) can be written as

$$H = H_0 + \frac{g}{\sqrt{2}}(Q + Q^+) - \frac{1}{2}g\left(J_z + \tfrac{1}{2} + \delta'\right)\sigma_z \tag{6}$$

where

$$H_0 = \tfrac{1}{2}g\left(\delta' + \tfrac{1}{2}\right) + \omega_P N + \left(E_{+-} + \tfrac{1}{2}g\right)J_z \qquad \delta' = \frac{\delta}{g}. \tag{7}$$

Using the commutation relation $[L_z, L_\pm] = \pm L_\pm$, we can prove that H_0 commutes with N', Q and Q^+, i.e.

$$[H_0, N'] = [H_0, Q] = [H_0, Q^+] = 0. \tag{8}$$

With the aid of the supersymmetric transformation generators defined above, we construct the supersymmetric unitary transformation operator so that the Hamiltonian in equation (6) can be diagonalized. The supersymmetric unitary transformation operator is defined as

$$T = \exp\left[-\frac{\theta}{2}N'^{-1/2}(Q^+ - Q)\right] \tag{9}$$

where $N'^{-1/2}$ is defined as

$$N'^{-1/2} = (L_+L_-)^{-1/2}\sigma_{--} + (L_-L_+)^{-1/2}\sigma_{++} \tag{10}$$

and θ is a function of operators to be determined later, and is supposed to satisfy the following commutation relations:

$$[\theta, Q] = \left[\theta, Q^+\right] = \left[N'^{-1/2}, \theta\right] = [\theta, H_0] = 0. \tag{11}$$

Therefore, equation (9) can be expanded into the following form:

$$T = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)N'^{-1/2}(Q^+ - Q). \tag{12}$$

From equations (8), (11) and (12), we have

$$T^{-1}H_0T = H_0 \tag{13}$$

$$T^{-1}(Q + Q^+)T = \cos(\theta)(Q + Q^+) + \sin(\theta)\sqrt{N'}\sigma_z. \tag{14}$$

$$T^{-1}\sigma_zT = \cos(\theta)\sigma_z - \sin(\theta)N'^{-1/2}(Q + Q^+). \tag{15}$$

Therefore,

$$\begin{aligned}H' = T^{-1}HT = H_0 &+ \frac{g}{\sqrt{2}}\cos(\theta)(Q + Q^+) + \frac{g}{\sqrt{2}}\sin(\theta)\sqrt{N'}\sigma_z \\ &-\tfrac{1}{2}g(J_z + \tfrac{1}{2} + \delta')[\cos(\theta)\sigma_z - \sin(\theta)N'^{-1/2}(Q + Q^+)].\end{aligned} \tag{16}$$

If we let

$$\cos(\theta) = \frac{J_z + \frac{1}{2} + \delta'}{\sqrt{2N' + \left(J_z + \frac{1}{2} + \delta'\right)^2}} \qquad \sin(\theta) = -\frac{\sqrt{2N'}}{\sqrt{2N' + \left(J_z + \frac{1}{2} + \delta'\right)^2}} \tag{17}$$

we can obtain the diagonalized Hamiltonian as follows:

$$\begin{aligned}H' = T^{-1}HT &= H_0 - \tfrac{1}{2}g\sqrt{2N' + \left(J_z + \tfrac{1}{2} + \delta'\right)^2}\sigma_z \\ &= H_0 - \frac{g}{\sqrt{2}}\sqrt{\left(L^2 - J_z^2 + \tfrac{1}{4}\right) + \tfrac{1}{2}\left(J_z + \tfrac{1}{2} + \delta'\right)^2}\sigma_z.\end{aligned} \tag{18}$$

It should be pointed out that equation (17) should be understood in the sense of eigenvalues and eigenvalue equations for the operators L^2, J_z. According to the theory of angular momentum, and using the formulae

$$L_+|N, l, m\rangle = \sqrt{l(l+1) - m(m+1)}\,|N, l, m+1\rangle \tag{19a}$$

$$L_-|N, l, m\rangle = \sqrt{l(l+1) - m(m-1)}\,|N, l, m-1\rangle \tag{19b}$$

where $|N, l, m\rangle$ is the common eigenstate of the operators N, L^2 and L_z, for a given N, $l = N, N-2, \ldots, [N - 2\,\mathrm{int}(N/2)]$, where $\mathrm{int}(N/2) = N/2$ for even N and $\mathrm{int}(N/2) = (N-1)/2$ for odd N, and for any allowable l, $m = 0, \pm 1, \pm 2, \ldots, \pm l$, we easily see that the eigenvectors of H' read

$$\left|\Psi_1'\right\rangle = |N, l, m, +\rangle \qquad \left|\Psi_2'\right\rangle = |N, l, m, -\rangle \tag{20}$$

where the two atomic levels $|\pm\rangle$ satisfy $\sigma_z|\pm\rangle = \pm|\pm\rangle$. Thus the eigenvalues and eigenvectors of H are given by, respectively,

$$E_1 = E_0(N, m) - \tfrac{1}{2}g\Omega(l, m) \tag{21a}$$

$$E_2 = E_0(N, m-1) + \tfrac{1}{2}g\Omega(l, m-1) \tag{21b}$$

$$|\Psi_1\rangle = T\left|\Psi_1'\right\rangle$$
$$= \sqrt{\frac{1}{2} + \frac{m+1+\delta'}{2\Omega(l,m)}}\,|N, l, m, +\rangle - \sqrt{\frac{1}{2} - \frac{m+1+\delta'}{2\Omega(l,m)}}\,|N, l, m+1, -\rangle \tag{21c}$$

$$|\Psi_2\rangle = T\left|\Psi_2'\right\rangle$$
$$= \sqrt{\frac{1}{2} + \frac{m+\delta'}{2\Omega(l,m-1)}}\,|N, l, m, -\rangle + \sqrt{\frac{1}{2} - \frac{m+\delta'}{2\Omega(l,m-1)}}\,|N, l, m-1, +\rangle \tag{21d}$$

where

$$E_0(N, m) = \tfrac{1}{2}g(m+1+\delta') + \omega_P N + E_{+-}\left(m + \tfrac{1}{2}\right) \tag{21e}$$

$$\Omega(l, m) = \sqrt{2l(l+1) - 2m(m+1) + (m+1+\delta')^2}. \tag{21f}$$

From equations (21), one easily sees that when $m = l$, $|\Psi_1\rangle = |N, l, l, +\rangle$ is a single state, the corresponding energy eigenvalue $E_1 = \omega_P N + E_{+-}(l + \frac{1}{2})$, and when $m = -l$, $|\Psi_2\rangle = |N, l, -l, -\rangle$ is also a single state, the corresponding energy eigenvalue $E_2 = \omega_P N - E_{+-}(l + \frac{1}{2}) + g(-l + \delta')$. It is interesting to see that these single states represent states of no coupling between the atom and the fields, which is quite similar to the ground state of the Jaynes–Cummings model.

Now, we discuss the time evolution of wavefunction from arbitrary initial conditions. Denote by $|\Psi(0)\rangle$ an arbitrary initial condition of the system:

$$|\Psi(0)\rangle = \sum_{n_P, n_S, n_A = 0}^{\infty} \left[C^{+}_{n_P, n_S, n_A}\,|n_P, n_S, n_A, +\rangle + C^{-}_{n_P, n_S, n_A}\,|n_P, n_S, n_A, -\rangle\right]. \tag{22}$$

In [5], Wu gave a formula expressing the eigenvectors of the orbital momentum $|N, l, m\rangle$ in terms of the Fock states $|n_P, n_S, n_A\rangle$

$$|N, l, m\rangle = \sum_k B_k \left|2k, \frac{N-m}{2} - k, \frac{N+m}{2} - k\right\rangle \tag{23a}$$

where $k = 0, 1, 2, \ldots, (N - |m|)/2$ for even $l + m$, and $k = \frac{1}{2}, \frac{3}{2}, \ldots, (N - |m|)/2$ for odd $l + m$,

$$B_k = 2^k \sqrt{\frac{(l+m)!(l-m)!}{(2l)!}}\, l!\, C_l \sum_{r=r_{\min}}^{r_{\max}} (-1)^r \frac{1}{4^r r!}$$
$$\times \frac{\sqrt{(2k)!\left(\frac{N-m}{2} - k\right)!\left(\frac{N+m}{2} - k\right)!}}{\left(\frac{N-l}{2} - r\right)!\left(\frac{l-m}{2} - k + r\right)!\left(\frac{l+m}{2} - k + r\right)!(2k - 2r)!} \tag{23b}$$

$$C_l = \left[\sum_{r=0}^{(N-l)/2} \frac{(2r)!\left(\frac{N+l}{2} - r\right)!}{4^r (r!)^2 \left(\frac{N-l}{2} - r\right)!}\right]^{-1/2} \tag{23c}$$

$$r_{\min} = \max[0, k - (l - |m|)/2)] \qquad r_{\max} = \min[k, (N-l)/2]. \tag{23d}$$

Similarly, we can expand the Fock states $|n_P, n_S, n_A\rangle$ in terms of the eigenvectors of the orbital momentum $|N, l, m\rangle$,

$$|n_P, n_S, n_A\rangle = \sum_l A_l |N, l, m\rangle \tag{24a}$$

where $N = n_P + n_S + n_A$, $m = n_A - n_S$, $l = N, N-2, \ldots, |m|$ for even $N+m$, and $l = N, N-2, \ldots, |m|+1$ for odd $N+m$,

$$A_l = 2^{n_P/2}\sqrt{\frac{(l+m)!(l-m)!}{(2l)!}}\, l! C_l \sum_{r=r_{\min}}^{r_{\max}} (-1)^r \frac{1}{4^r r!} \times \frac{\sqrt{n_P! n_S! n_A!}}{\left(\frac{N-l}{2}-r\right)!\left(\frac{l-m-n_P}{2}+r\right)!\left(\frac{l+m-n_P}{2}+r\right)!(n_P-2r)!} \tag{24b}$$

$$r_{\min} = \max[0, (n_P - l + |m|)/2)] \qquad r_{\max} = \min[n_P/2, (N-l)/2]. \tag{24c}$$

By means of equation (24a), $|\Psi(0)\rangle$ can be written as

$$|\Psi(0)\rangle = \sum_{n_P, n_S, n_A, l} \left[A_l C^+_{n_P,n_S,n_A} |N, l, m, +\rangle + A_l C^-_{n_P,n_S,n_A} |N, l, m, -\rangle\right]. \tag{25}$$

With the help of equations (12), (18) and (21), we obtain the wavefunction $|\Psi(t)\rangle$

$$\begin{aligned}|\Psi(t)\rangle &= \exp(-\mathrm{i}Ht)|\Psi(0)\rangle = T\exp(-\mathrm{i}H't)T^{-1}|\Psi(0)\rangle \\ &= \sum_{n_P,n_S,n_A,l} \{A_l C^+_{n_P,n_S,n_A} \exp[-\mathrm{i}E_0(N,m)t] \\ &\quad \times[(\cos g\Omega(l,m)t/2 + \mathrm{i}\cos\theta_1 \sin g\Omega(l,m)t/2)|N,l,m,+\rangle \\ &\quad + \mathrm{i}\sin\theta_1 \sin g\Omega(l,m)t/2|N,l,m+1,-\rangle] \\ &\quad + A_l C^-_{n_P,n_S,n_A} \exp[-\mathrm{i}E_0(N,m-1)t] \\ &\quad \times[(\cos g\Omega(l,m-1)t/2 - \mathrm{i}\cos\theta_2 \sin g\Omega(l,m-1)t/2)|N,l,m,-\rangle \\ &\quad + \mathrm{i}\sin\theta_2 \sin g\Omega(l,m-1)t/2|N,l,m-1,+\rangle]\}\end{aligned} \tag{26a}$$

where

$$\begin{aligned}\cos\theta_1 &= \frac{m+1+\delta'}{\Omega(l,m)} \qquad & \sin\theta_1 &= -\frac{\sqrt{2l(l+1)-2m(m+1)}}{\Omega(l,m)} \\ \cos\theta_2 &= \frac{m+\delta'}{\Omega(l,m-1)} \qquad & \sin\theta_2 &= -\frac{\sqrt{2l(l+1)-2m(m-1)}}{\Omega(l,m-1)}.\end{aligned} \tag{26b}$$

These general results immediately give the solution to any specific cases. For example, we suppose that at time $t = 0$, pump mode is vacuum, Stokes and anti-Stokes mode are two-mode $SU(1,1)$ coherent state, and the atom is in an excited state $|+\rangle$,

$$|\Psi(0)\rangle = \sum_{n=0}^{\infty} F_n\,|0, n+q, n, +\rangle = \sum_{n=0}^{\infty}\sum_{\lambda=0}^{n} F_n A_\lambda\,|2n+q, q+2\lambda, -q, +\rangle \tag{27a}$$

where

$$F_n = (1-\xi^*\xi)^{(1+q)/2}\sqrt{\frac{(n+q)!}{n!q!}}\,\xi^n \tag{27b}$$

$$A_\lambda = \sqrt{\frac{(2\lambda)!(2q+2\lambda)!(n+q)!n!}{(2q+4\lambda)!}}\,\frac{(q+2\lambda)!C_\lambda}{(n-\lambda)!(q+\lambda)!\lambda!} \tag{27c}$$

$$C_\lambda = \left[\sum_{r=0}^{n-\lambda} \frac{(2r)!(n+q+\lambda-r)!}{4^r (r!)^2 (n-\lambda-r)!}\right]^{-1/2}. \tag{27d}$$

From equations (26) we obtain the time evolution of the wavefunction

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}\sum_{\lambda=0}^{n} F_n A_\lambda \exp(-\mathrm{i}E_0 t)\{\mathrm{i}\sin\theta_0 \sin g\Omega t/2|2n+q, q+2\lambda, -q+1, -\rangle$$
$$+[\cos g\Omega t/2 + \mathrm{i}\cos\theta_0 \sin g\Omega t/2]|2n+q, q+2\lambda, -q, +\rangle\} \tag{28a}$$

where

$$E_0 = \tfrac{1}{2}g(\delta' + 1 - q) + \omega_P(2n+q) - E_{+-}(q - \tfrac{1}{2}) \tag{28b}$$

$$\Omega = \sqrt{2}\sqrt{(q+2\lambda)(q+1+2\lambda) - q(q-1) + (\delta'+1-q)^2} \tag{28c}$$

$$\cos\theta_0 = \frac{\delta'+1-q}{\Omega} \qquad \sin\theta_0 = -\frac{\sqrt{2}\sqrt{(q+2\lambda)(q+1+2\lambda) - q(q-1)}}{\Omega}. \tag{28d}$$

It is worth pointing out that if $\delta' = q - 1$, the Rabi frequency Ω given by equation (28c) is the same as one without a Stark term and frequency detuning.

In summary, based on supersymmetric quantum mechanics theory, we have introduced a supersymmetric unitary transformation to diagonalize the Hamiltonian of coupled-channel cavity QED model which include the Stark term and the frequency detuning. We have obtained its eigenvalue, eigenstates and time evolution of the state vector. On one hand, these general results immediately give the solution to any specific cases, and will facilitate the subsequent investigations of the dynamical and statistical properties of the system, on the other hand, this method is not only simple but universal as well.

References

[1] de Lange O L and Raab E R 1991 Operator methods in quantum mechanics (New York: Oxford University Press) p 71

[2] Cooper F, Khare A and Sukhateme U 1995 *Phys. Rev.* **251** 267

[3] Dutt R, Khare A and Sukhateme U 1988 *Am. J. Phys.* **56** 163

[4] Fan H Y and Li L S 1997 *Representation and Transformation Theory in Quantum Mechanics—Progress of Dirac's Symbolic Method* (Shanghai: Scientific and Technical Publishers) p 181 (in Chinese)

[5] Wu Y 1996 *Phys. Rev.* A **54** 4534

[6] Wang L, Puri R R and Eberly J H 1992 *Phys. Rev.* A **46** 7192